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CLOSED FORM SOLUTION TO CONTROL EQUATIONS

[6]

By

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

GEORGE C. MARSHALL SPACE FLIGHT CENTER

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ABSTRACT

The paper shows how one may avoid the unreasonably small step-size required for the numerical integration of some systems of linear ordinary differential equations with nearly constant coefficients. The non-homogeneous parts of the equations are represented by polynomials over short time intervals, and closed form expressions for the integrals on these time intervals are obtained in terms of the polynomial coefficients.

The paper is particularly concerned with equations representing electrical networks utilized in missile control systems.

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ACKNOWLEDGEMENTS

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LIST OF SYMBOLS

A	an n x n matrix $\left(\dot{\bar{q}} = A\bar{q} + \bar{b}E_{in}\right)$
$\bar{b}$	an n x 1 vector
$\bar{c}$	$\bar{c} = P^{-1}\bar{b}$
D	diagonal matrix of eigenvalues
E_{in}	exciting function
E_{out}	filter response
$\bar{h}$	the i th component of $\bar{h}$ is $h_i = v_i p_i$
m	order of interpolation on E_{in}
n	order of A
P	transformation matrix
P^{-1}	inverse of P
$\bar{p}$	$\bar{p} = P^{-1}\bar{q}$
$\bar{q}$	the n x 1 vector of dependent variables
r_k	coefficient of t^k in interpolating polynomial
t	independent variable (time)
T	time at beginning of interval over which one wishes to integrate
Δt	duration of time intervals over which closed form integrations are to be performed
$\bar{u}$	an n x 1 vector
$\bar{u}^T$	transpose of u, defined so that $E_{out} = \bar{u}^T\bar{q}$
$\bar{v}$	$\bar{v}^T = \bar{u}^T P$
$\bar{x}_i$	a right eigenvector of A corresponding to λ_i (also the i th column of P)

$\bar{y}_i^T$	a left eigenvector of A corresponding to λ_i (also the i th row of P^{-1})
γ_i	$\gamma_i = (\bar{u}^T \bar{x}_i)\ (\bar{y}_i \bar{b})$
λ_i	i th eigenvalue of A ($i = 1, 2, \ldots, n$)
$\cong$	approximately equal to

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SUMMARY

The paper shows how one may avoid the unreasonably small step-size required for the numerical integration of some systems of linear ordinary differential equations with nearly constant coefficients. The non-homogeneous parts of the equations are represented by polynomials over short time intervals, and closed form expressions for the integrals on these time intervals are obtained in terms of the polynomial coefficients.

The paper is particularly concerned with equations representing electrical networks utilized in missile control systems.

SECTION I. INTRODUCTION

The control systems of many missiles utilize electrical networks known as filters. A filter can be used to delay, amplify, or modify in some other manner signals given to it by means of an electromotive force E_{in}. The modified signal E_{out} is also an electromotive force. Such filters can be represented mathematically by means of systems of first order ordinary linear differential equations. However, the numerical integration of these equations often requires an unreasonably long time, even on a high speed digital computer. The problem is particularly frustrating because of the hope that the trajectory and control equations of a missile may be integrated on digital computers in real time or simultaneously with the flight of the missile.

The main purpose of this paper is to propose a solution which is now in use at the Marshall Space Flight Center. Actually, the method is applicable to any system of ordinary linear differential equations.

The precedure may be applied to find E_{out} at time $T + \Delta t$ in terms of conditions given at time T whenever it is possible to accurately represent E_{in} as some function, such as a polynomial or trigonometric function, over the interval from T to $T + \Delta t$. When it is possible to represent E_{in} in such a manner, the linear differential equations may be solved in closed form over the interval, and the method presented herein is a simple and concise technique for doing just that. A similar procedure was proposed in reference [1], but it was developed only for the most simple filters.

The appendix briefly discusses the origin of the numerical difficulties encountered with linear differential equations. It also uses the analysis given in the previous sections to show how these difficulties may be removed in many cases in order to allow for numerical integration at an acceptable step-size.

SECTION II. A PRECISE STATEMENT OF THE PROBLEM

The problem under consideration is that of solving the system

$$\left\{ \begin{array}{l} \dot{\bar{q}} = A\,\bar{q} + E_{in}\,\bar{b} \\ E_{out} = \bar{u}^{\,T}\,\bar{q} \end{array} \right\} \qquad (1)$$

in closed form at time $T + \Delta t$ in terms of conditions at time T when given that E_{in} is a polynomial

$$\sum_{k=0}^{m} r_k t^k$$

over the interval

$$T \leq t \leq T + \Delta t.$$

Here $\bar{q}$, $\bar{b}$, and $\bar{u}$ represent n x 1 matrices and A represents an n x n matrix. The symbol $\bar{u}^{\,T}$ stands for the transpose of $\bar{u}$.

We assume that A, $\bar{b}$, and $\bar{u}$ are known and that $\bar{q}$ has the dependent variables as its components. Furthermore, it is assumed that A is similar to a diagonal matrix, i.e., that there exists a non-singular matrix P such that

$$P^{-1} AP = D ,$$

where D is a diagonal matrix having the eigenvalue λ_i of A in its i th diagonal position for i = 1, 2, ..., n. The latter is true for any matrix whose eigenvalues are all distinct, as well as for some other matrices.

The i th column $\bar{x}_i$ of P is necessarily a right eigenvector corresponding to λ_i. The i th row of $\bar{y}_i^T$ of P^{-1} is a left eigenvector.

SECTION III. A SIMPLIFYING TRANSFORMATION

We define a vector $\bar{p}$ by means of the equation

$$\bar{p} = P^{-1} \bar{q}$$

Substitution of $\bar{q} = P \bar{p}$ into (1) yields

$$\left\{ \begin{array}{l} P \dot{\bar{p}} = A (P \bar{p}) + E_{in} \bar{b} \\ E_{out} = \bar{u}^T (P \bar{p}) \end{array} \right\}$$

$$\left\{ \begin{array}{l} \dot{\bar{p}} = (P^{-1} A P) \bar{p} + E_{in} (P^{-1} \bar{b}) \\ E_{out} = (\bar{u}^T P) \bar{p} \end{array} \right\}$$

$$\left\{ \begin{array}{l} \dot{\bar{p}} = D \bar{p} + E_{in} \bar{c} \\ E_{out} = \bar{v}^T \bar{p} \end{array} \right\}$$

where

$$\bar{c} = P^{-1} \bar{b}, \quad \bar{v}^T = \bar{u}^T P .$$

Letting p_i, c_i, and v_i represent the i th components of $\bar{p}$, $\bar{c}$, and $\bar{v}$, respectively, we have

$$\left\{ \begin{array}{l} \dot{p}_i = \lambda_i p_i + c_i E_{in} \\ E_{out} = \sum_{i=1}^{n} v_i p_i \end{array} \right\} \qquad (i = 1, \ldots, n)$$

$$\left\{ \begin{array}{l} \dot{h}_i = \lambda_i h_i + \gamma_i E_{in} \\ E_{out} = \sum_{i=1}^{n} h_i \end{array} \right\} \qquad (i = 1, \ldots, n) \qquad (2)$$

where

$$h_i = v_i p_i$$

and

$$\gamma_i = v_i c_i = (\bar{u}^T \bar{x}_i)(\bar{y}_i^T \bar{b})$$

for $i = 1, \ldots, n$.

The initial value of h_i is zero if $\bar{q} = 0$ initially as is usually the case.

SECTION IV. A CLOSED FORM SOLUTION

The standard solution for the equation

$$\dot{h}_i = \lambda_i h_i + \gamma_i E_{in}$$

is

$$h_i = e^{\int \lambda_i dt} \int e^{-\int \lambda_i dt} \gamma_i E_{in}\, dt + C_i$$

where C_i is the constant of integration.

Therefore,

$$h_i (T + \Delta t) = e^{\lambda_i \Delta t} \left[\gamma_i \int_T^{T+\Delta t} e^{-\lambda_i (t-T)} E_{in}(t)\, dt + h_i (T) \right]$$

Before integrating further, we are forced to assume some form for $E_{in}(t)$. We assume that

$$E_{in}(t) = \sum_{k=0}^{m} r_k (t-T)^k$$

over the interval

$$T \leq t \leq T + \Delta t.$$

Thus,

$$\int_T^{T+\Delta t} e^{-\lambda_i (t-T)} E_{in}\, dt = \sum_{k=0}^{m} r_k \int_T^{T+\Delta t} (t-T)^k e^{-\lambda_i (t-T)}\, dt$$

$$\int_T^{T+\Delta t} e^{-\lambda_i (t-T)} E_{in}\, dt = \sum_{k=0}^{m} r_k \int_0^{\Delta t} \tau^k e^{-\lambda_i \tau}\, d\tau \ ,$$

where $\tau = t - T$. If $\lambda_i \neq 0$, repeated integration by parts yields

$$\int \tau^k e^{-\lambda_i}\, d\tau = -\frac{e^{-\lambda_i \tau}}{\lambda_i^{k+1}} \sum_{s=0}^{k} \frac{k!}{s!} (\lambda_i \tau)^s \ .$$

A few straightforward manipulations give us

$$h_i (T+\Delta t) = h_i (T) \cdot e^{\lambda_i \Delta t} + \gamma_i \sum_{k=0}^{m} \frac{k!}{\lambda_i^{k+1}} \left[e^{\lambda_i \Delta t} - \sum_{s=0}^{k} \frac{(\lambda_i \Delta t)^s}{s!} \right] r_k$$

If $\lambda_i \Delta t$ is small, the quantity

$$\frac{k!}{\lambda_i^{k+1}} \left[e^{\lambda_i \Delta t} - \sum_{s=0}^{k} \frac{(\lambda_i \Delta t)^s}{s!} \right] \tag{3}$$

can be computed by means of the series

$$k! \, (\Delta t)^{k+1} \sum_{s=0}^{\infty} \frac{(\lambda_i \Delta t)^s}{(s+k+1)!} \tag{4}$$

truncated after several terms, thereby avoiding subtraction of a number from another nearly equal to it and, consequently, loss of perhaps several significant figures. Observe that if $\lambda_i = 0$, (4) becomes

$$\frac{(\Delta t)^{k+1}}{k+1}$$

which is exactly what one should obtain in the place of (3) for the case in which $\lambda_i = 0$. Therefore, (4) is applicable to the case in which $\lambda_i = 0$.

SECTION V. SUMMARY

In summary,

$$\left\{ \begin{array}{l} h_i (T + \Delta t) = h_i (T) \cdot e^{\lambda_i \Delta t} + \sum_{k=0}^{m} f_{ik} r_k \\ \\ E_{out} (T + \Delta t) = \sum_{i=1}^{n} h_i (T + \Delta t) \ , \end{array} \right\} \qquad (5)$$

where

$$f_{ik} = \gamma_i \frac{k!}{\lambda_i^{k+1}} \left[e^{\lambda_i \Delta t} - \sum_{s=0}^{k} \frac{(\lambda_i \Delta t)^s}{s!} \right] \quad (\lambda_i \neq 0)$$

$$f_{ik} = \gamma_i \frac{(\Delta t)^{k+1}}{k+1} \qquad (\lambda_i = 0)$$

$$\gamma_i = (\bar{u}^T \bar{x}_i) (\bar{y}_i^T \bar{b})$$

and $\bar{x}_i$ and $\bar{y}_i^T$ are right and left eigenvectors of A corresponding to λ_i and scaled in such a manner that

$$\bar{y}_i^T \bar{x}_i = 1 \ .$$

Equations (5) give the desired closed form solution which expresses $E_{out} (T + \Delta t)$ in terms of conditions at time T. If the value of Δt is changed during the process of integration, the coefficients

$$e^{\lambda_i \Delta t}, \ f_{ik} \qquad (i = 1, \ldots, n; \ k = 1, \ldots, m)$$

must be recomputed.

On examination of (5), it may be seen that the parameters

$$\lambda_i, \ \gamma_i \qquad (i = 1, \ldots, n) \qquad (6)$$

completely determine the responses of the filter, assuming a particular function E_{in} and particular initial conditions are given. If the filter is modified, then the parameters (6) will change (theoretically). For example if the elements of A, $\bar{b}$, and $\bar{u}$ vary with time, then λ_i and γ_i will vary with time. (In deriving the equations (5), it was assumed that, for practical purposes, λ_i and γ_i are constant over the time interval from T to T + Δt.)

It should be observed that λ_i, γ_i, h_i, and f_{ik} may be complex, but in practice the computations may be easily handled in real arithmetic if one works with pairs of complex conjugates in the appropriate manner.

In passing, we note that if the polynomial coefficients r_k (k = 0, 1, ... , m) are expressed as linear combinations of

$$E_{in}\ (t_0 + k\Delta t) \qquad (k = 0, 1, \ldots, m)$$

for some value t_0 of time, then the first of equations (5) may be rewritten as

$$h_i\ (T+\Delta t) = h_i \cdot\ (T) \cdot e^{\lambda_i \Delta t} + \sum_{k=0}^{m} g_{ik} \cdot E_{in}\ (t_0 + k\Delta t)$$

for some constants g_{ik}.

SECTION VI. AN EXAMPLE

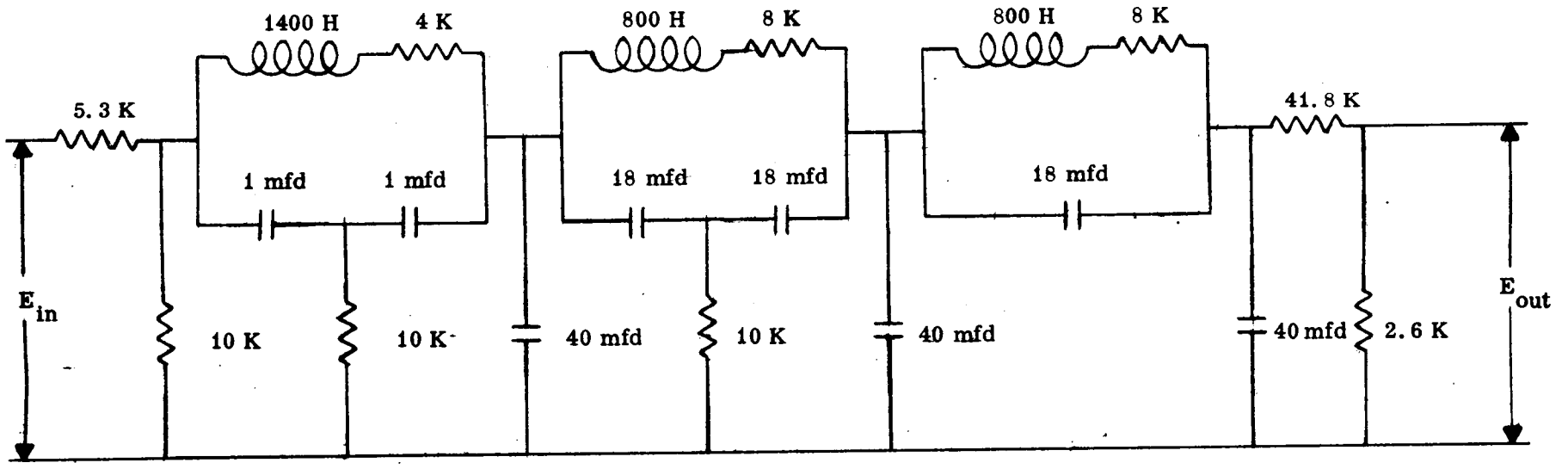
The filter shown in Figure 1 was represented by means of nine linear first order differential equations. Using

$$E_{in} = \sin\ (2\pi t)\ ,$$

E_{out} was determined by the closed form method described in this paper.

Linear interpolation was used to represent E_{in} on each time interval employed. All initial conditions were taken to be zero. Time increments of $\Delta t = .2$ sec., $\Delta t = .06$ sec., and $\Delta t = .02$ sec. were utilized.

For comparison, the integration was also performed by the fourth order Runge-Kutta method using $\Delta t = .006$ sec. The latter technique diverged when $\Delta t = .007$ sec. was employed.



H ≡ Henries, K ≡ kilo-ohms, mfd ≡ micro-farads

FIGURE 1. FILTER CIRCUIT USED IN THE EXAMPLE

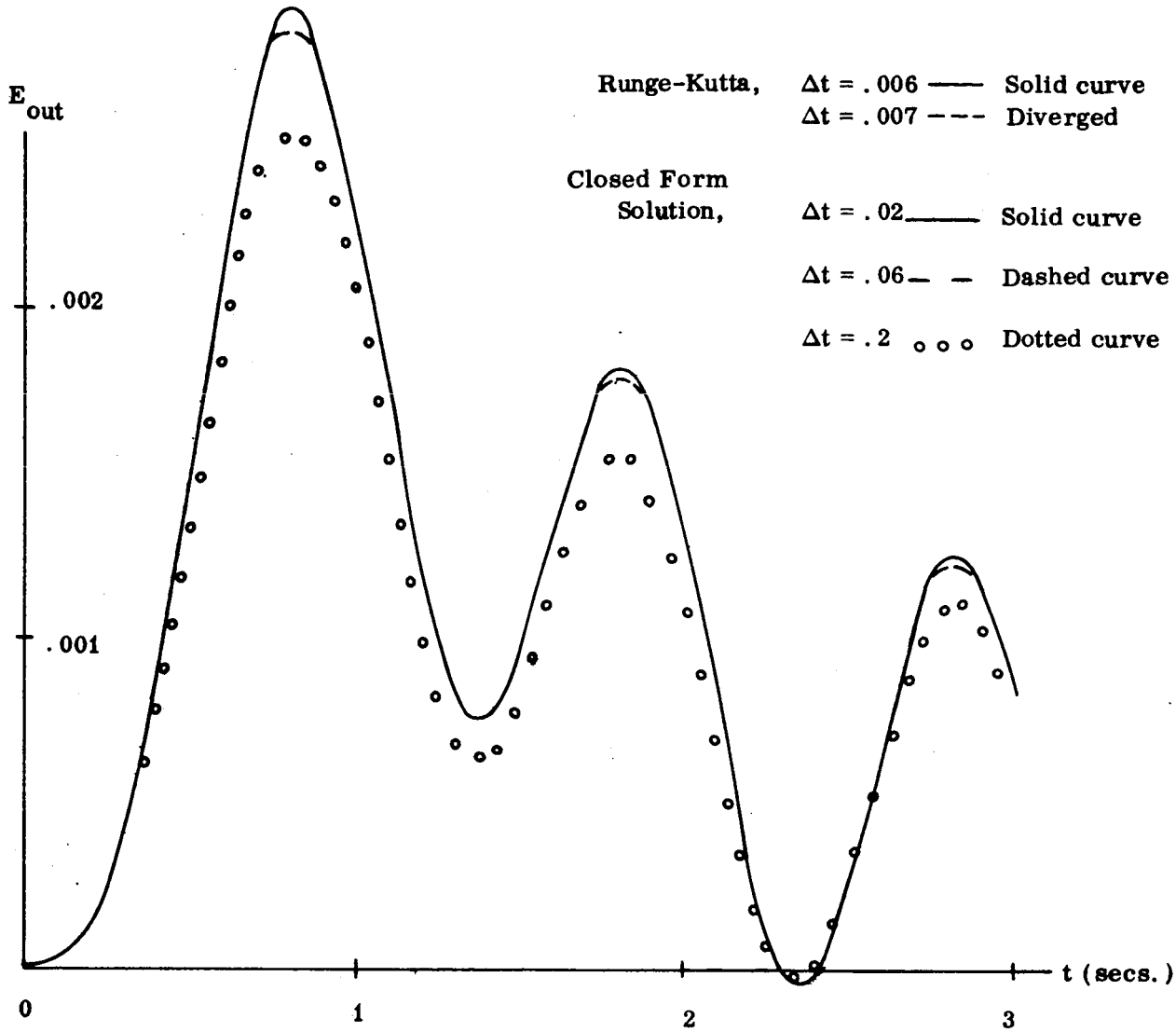


FIGURE 2. INTEGRATION RESULTS FOR $E_{in} = \sin(2\pi t)$

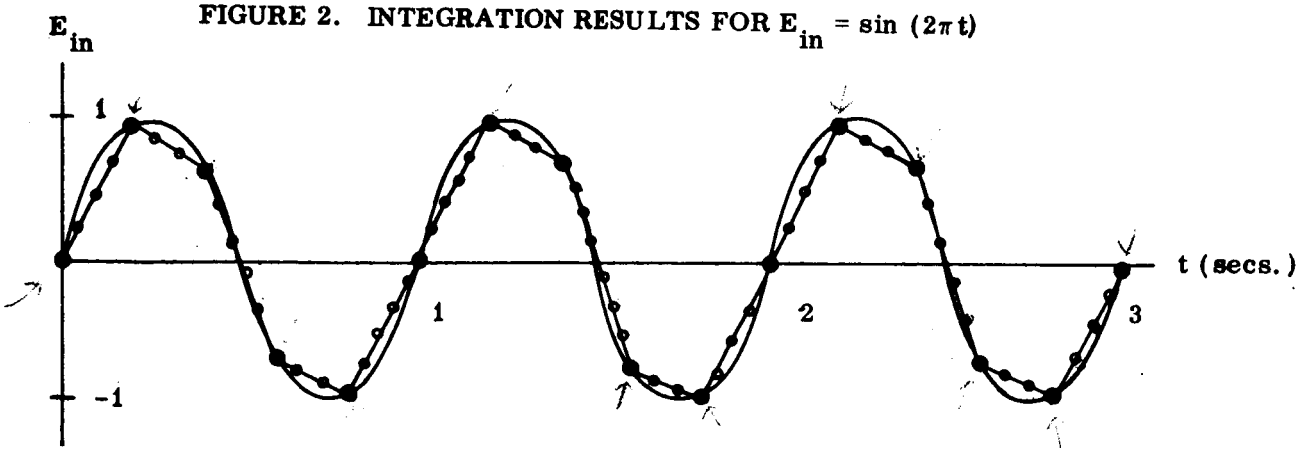


FIGURE 3. EXCITING FUNCTION: Sin $(2\pi t)$ REPRESENTED BY LINE SEGMENTS.

The integration results are shown in Figure 2. In Figure 3, we show the results of representation of sin $(2\pi t)$ by line segments over the $\Delta t = .2$ sec. intervals. The difference between the solid and dashed curves in Figure 2 is accounted for primarily by the difference between the curves in Figure 3.

SECTION VII. APPENDIX

The analysis presented in this paper leads to a method that in many cases may be used to alter a system of linear differential equations in a manner which will not significantly affect the solution but will enable one to considerably enlarge the step-size required for numerical integration. If one lets $E_{in} = \sin(\omega t)$, it can be shown that

$$h_i(t) = \frac{\gamma_i}{\sqrt{\lambda_i^2 + \omega^2}} \left[\sin(\omega t + \theta_i) - e^{\lambda_i t} \sin \theta_i \right]$$

provided λ_i is real, $h_i(0) = 0$, and

$$\theta_i = \tan^{-1}(\omega/\lambda_i) .$$

The largest eigenvalue in magnitude of the example of the preceding section is equal to about -640, whereas the smallest is approximately -.78. Call these eigenvalues λ_1 and λ_2, respectively. The parameters γ_1 and γ_2 in this case are

$$\gamma_1 = .012, \quad \gamma_2 = .028 .$$

If $\omega \cong 2\pi$, then it is easily verified that the contribution of h_1 in (5) to E_{out} is small compared to that of h_2. In fact, the contribution of h_1 does not quite show up in Figure 2.

The h_1 transient term

$$w = a_1 e^{\lambda_1 t} ,$$

where a_1 is a constant, is even of less significance, but this is the very term which gives the most trouble to numerical integration techniques (see [2]);

even though w is small, its m th derivative $\lambda_i^m w$ can be very large and, consequently, very troublesome.

Now, returning to the general problem, let use suppose that h_i is negligibly small for $i = 1, \ldots, N$ and that $\bar{h}(0) = 0$ (i.e., that $\bar{q}(0) = 0$). We have shown that

$$\left\{ \begin{array}{l} \dot{h}_i = \lambda_i h_i + \gamma_i E_{in} \\ E_{out} = \sum_{i=1}^{n} h_i \end{array} \right\} \qquad (i = 1, \ldots, n) \tag{7}$$

If we set λ_i and γ_i equal to zero for $i = 1, \ldots, N$, then $h_i(t) \equiv 0$ for $i = 1, \ldots, N$. Consequently, the system (7) with

$$\lambda_i = \gamma_i = 0 \qquad (i = 1, \ldots, N)$$

will have nearly the same solution as the original system (1). This new system is exactly what is obtained by replacing (1) with

$$\dot{\bar{q}} = A_0 \bar{q} + E_{in} \bar{b}_0 , \tag{8}$$

where

$$A_0 = A - \sum_{i=1}^{N} \lambda_i V_i ,$$

$$\bar{b}_0 = \left(I - \sum_{i=1}^{N} V_i \right) \bar{b} ,$$

$$V_i = \bar{x}_i \bar{y}_i^T$$

It can be easily shown that A_0 has the same eigenvalues and eigenvectors as A with the exception that $\lambda_1, \ldots, \lambda_N$ are replaced with zeros (as desired). On the other hand, $\bar{b}_0$ satisfies the equations:

$$\bar{y}_i^T \bar{b}_0 = 0 \qquad (i = 1, \ldots, N)$$

$$\bar{y}_i^T \bar{b}_0 = \bar{y}_i^T \bar{b} \qquad (i = N + 1, \ldots, n) .$$

This means that the parameters γ_i computed in terms of $\bar{b}_0$ rather than $\bar{b}$, are left unchanged for $i = N + 1, \ldots, n$ and are zero (as desired) for $i = 1, \ldots, N$.

Using fourth order Runge - Kutta and the problem of the preceding section, the system (8) was integrated after removal of

Case I the eigenvalue -640 of largest magnitude, and

Case II the two eigenvalues, -640 and -46, of largest magnitudes.

In Case I a step-size of .085 sec. led to close agreement with the correct solution. (Using .09 sec. there was no agreement.) In Case II a step-size of .1 sec. gave fairly close agreement with the correct solution.

In summary, using (8), it is often (perhaps always in practical problems) possible to substantially increase the step-size required for numerical integration of (1) in those cases in which there is at least one eigenvalue considerably larger in magnitude than the eigenvalue smallest in magnitude.

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the Marshall Space Flight Center Security Classification Officer. This report, in its entirety, has been determined to be Unclassified.

This document has also been reviewed and approved for technical accuracy.

DR. H. HOELZER
Director, Computation Division

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